The Ingram Conjecture

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Abstract

We prove the Ingram Conjecture, *i.e.,* we show that the inverse limit spaces of every two tent maps with different slopes in the interval $[1, 2]$ are non-homeomorphic. Based on the structure obtained from the proof, we also show that every self-homeomorphism of the inverse limit space of the tent map is pseudo-isotopic, on the core, to some power of the shift homeomorphism.

1   Introduction

Apart from their interest within continuum theory, inverse limit spaces play a key role in the description of uniformly hyperbolic attractors [25, 26], global 'Hénon-like' strange attractors [6] and the structure emerging from homoclinic tangencies in dynamical systems [4]. They find further use in the area of (substitution) tiling spaces [1] which, in some cases, are covering spaces of the type of inverse limit spaces with which we are concerned with in this paper; namely, those with a single tent map $T_s : [0, 1] \to [0, 1]$, $x \mapsto \min\{sx, s(1 - x)\}$ as bonding map. Such inverse limit spaces can be embedded in the plane as global attractors of homeomorphisms [19, 22, 12] and immersed in the plane as global attractors of skew product maps [15].

Inverse limit spaces are notoriously difficult to classify. In this paper, we solve in the affirmative the classification problem known as the Ingram Conjecture:

Theorem 1.1 (Ingram Conjecture). If $1 \le s < s' \le 2$, then the corresponding inverse limit spaces $\varprojlim([0,1], T_s)$ and $\varprojlim([0,1], T_{s'})$ are non-homeomorphic.

This is the main outstanding conjecture regarding dynamics on continua, dating back to at least the early nineties. In the "Continua with the Houston problem book" in 1995 [16, page 257], Ingram writes

> The [...] question was asked of the author by Stu Baldwin at the summer meeting of the AMS at Orono, Maine, in 1991... There is a related question which the author has considered to be of interest for several years. He posed it at a problem session at the 1992 Spring Topology Conference in Charlotte for the special case (*that the critical point has period*) $n = 5$.

It is clear that if two interval maps are topologically conjugate, then their inverse limit spaces are homeomorphic. Thus it may be more natural to ask the question for the 'fuller' logistic family $f_a(x) = ax(1-x)$, $a \in [0,4]$. It is well-known [18] that each logistic map is semi-conjugate to a tent-map T_s with $s = \exp(h_{top}(f_a))$ (provided $h_{top}(f_a) > 0$), but the logistic maps contain infinitely renormalizable maps as well as maps with periodic attractors, phenomena that are ruled out in the (uniformly expanding) setting of tent maps with slope $s > 1$. The effect of renormalization (*i.e.,* the existence of periodic intervals for period > 1) on the structure of the inverse limit space is well-understood, see [5]: it produces proper subcontinua that are periodic under the shift homeomorphism, that are homeomorphic with the inverse limit space of the renormalized map. Therefore the solution of the Ingram Conjecture also shows that every pair of logistic maps that are non-conjugate on their non-wandering sets have non-homeomorphic inverse limit spaces.

There have been several partial results to the Ingram Conjecture, *e.g.* Barge and Diamond [3], which solved the period $n = 5$ case, and [24, 14]. Complete solutions were obtained when the critical point is periodic by Kailhofer [17] (see also [8]), or has finite orbit by Štimac [23]. More recently, the case where the critical point is non-recurrent was solved in [21]. Further results that classify certain features of inverse limit spaces of tent maps with non-periodic recurrent critical orbits were obtained in *e.g.* [11, 20, 13].

Our solution to the Ingram Conjecture gives more information about the set of self-homeomorphisms on $\varprojlim([0,1], T_s)$: we show that any such homeomorphism behaves like an iterate of the shift homeomorphism σ.

The *critical point* $\frac{1}{2}$ of T_s is denoted by c, and we write $c_i = T^i(c)$. Although T_s is defined on $[0,1]$, there is a forward invariant interval $[c_2, c_1] = [s(1 - s/2), s/2]$, called

the *core*, on which T_s is surjective. We call $\varprojlim([c_2, c_1], T_s)$ the *core of the inverse limit space*. The space $\varprojlim([0, 1], T_s)$ is the union of the core of the inverse limit and a ray $\mathfrak{C}$ converging onto it.

Recall that the *composant* of $x \in X$ is defined as the union of all proper subcontinua of X containing x. For $1 \leq s < 2$, $\varprojlim([0, 1], T_s)$ has only two composants: $\mathfrak{C}$ and $\varprojlim([0, 1], T_s) \setminus \{(\dots, 0, 0, 0)\}$. But for $s > \sqrt{2}$, $\varprojlim([c_2, c_1], T_s)$ is indecomposable and hence has uncountably many pairwise disjoint composants, each of which is dense. If $s > \sqrt{2}$ and the orbit of c is finite, the composants of $\varprojlim([c_2, c_1], T_s)$ are the same as the arc-components. Otherwise, the composants can be very complicated. For $1 < s \leq \sqrt{2}$, the core has just two composants that overlap in a single arc-component.

Theorem 1.2. Given $s \in [1, 2]$, for every homeomorphism $h : \varprojlim([0, 1], T_s) \circlearrowleft$, there is an $R \in \mathbb{Z}$ such that h, restricted to the core $\varprojlim([c_2, c_1], T_s)$, is pseudo-isotopic to σ^R, *i.e.,* it permutes the composants of the core of the inverse limit in the same way as σ^R.

Our proof of the Ingram Conjecture relies on the properties of so-called link-symmetric arcs in the composant $\mathfrak{C}$ of $\varprojlim([0, 1], T_s)$ containing the endpoint $\alpha := (\dots, 0, 0, 0)$. Inverse limit spaces are chainable, and w.r.t. natural chains, a homeomorphism $h : \varprojlim([0, 1], T_{s'}) \to \varprojlim([0, 1], T_s)$ maps link-symmetric arc to link-symmetric arcs. From this we derive that maximal link-symmetric arcs in $\varprojlim([0, 1], T_{s'})$ centered at so-called snappy points s_i' map to link-symmetric arcs centered at snappy points $s_{i+M} \in \varprojlim([0, 1], T_s)$ for some $M \in \mathbb{Z}$ and all sufficiently large $i \in \mathbb{N}$.

This in turn implies that h maps so-called q-points close to p-points, while 'translating' their levels by a fixed number M. This shows that h effectively fixes the folding pattern of the zero-composant, with the Ingram Conjecture as an easy consequence. Additional arguments show that every self-homeomorphism of $\varprojlim([0, 1], T_s)$, when restricted to the core, is *pseudo-isotopic* to a power σ^R of the shift for some $R \in \mathbb{Z}$.

We give the basic definitions in the next section. In Section 3 we investigate the lengths of maximal link-symmetric arcs, leading in Section 4 to the proof that a homeomorphism between two unimodal inverse limit spaces induces a shift of indices of snappy points, and more generally, acts as a shift on the levels of q-points and p-points. This leads to the proof of the Ingram Conjecture. Finally, in Section 5, we prove the remaining results on pseudo-isotopy.

2 Definitions

Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Let $T_s : [0, 1] \to [0, s/2]$, $T_s(x) = \min\{sx, s(1-x)\}$ be the tent map with slope $s \in [1, 2]$ and critical point $c = \frac{1}{2}$. Write $c_i = c_i(s) := T_s^i(c)$, so in particular $c_1 = \frac{s}{2}$ and $c_2 = s(1 - \frac{s}{2})$.

The inverse limit space $\varprojlim([0, 1], T_s)$ is the collection of backward orbits

$$\{x = (\ldots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \leq 0\},$$

equipped with metric $d(x, y) = \sum_{n \leq 0} 2^n |x_n - y_n|$ and *induced (or shift) homeomorphism* $\sigma = \sigma_s$ given by

$$\sigma(\ldots, x_{-2}, x_{-1}, x_0) = (\ldots, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

Let $\pi_p : \varprojlim([0, 1], T_s) \to [0, 1]$, $\pi_p(x) = x_{-p}$, be the p-th projection map. Since T_s fixes 0, $\varprojlim([0, 1], T_s)$ contains the endpoint $\alpha := (\ldots, 0, 0, 0)$. The composant of $\varprojlim([0, 1], T_s)$ containing this point is denoted by $\mathfrak{C}$; it is a ray converging from α to, but disjoint from, the core of the inverse limit space $\varprojlim([c_2, c_1], T_s)$.

Frequently, the Ingram Conjecture is posed for slopes $s, s' \in [\sqrt{2}, 2]$ only, because for $0 < s \leq \sqrt{2}$, $\varprojlim([c_2, c_1], T_s)$ is decomposable. Since $\varprojlim([0, 1], T_s)$ is a single point for $s = 0$ and a single arc for $s \in (0, 1]$, we will always assume that all slopes s are greater than 1. The next two lemmas show how to reduce the case $s \in (1, \sqrt{2}]$ to $s \in (\sqrt{2}, 2]$.

Lemma 2.1. For $2^{1/2^{n+1}} \leq s \leq 2^{1/2^n}$, $n \in \mathbb{N}$, the core of the inverse limit space $\varprojlim([c_2, c_1], T_s)$ is homeomorphic with two copies of $\varprojlim([0, 1], T_{s^2})$ joined at their endpoints.

Proof. For this range of s, $T_s([c_2, p]) = [p, c_1]$ and $T_s([p, c_1]) = [c_2, p])$, where $p := \frac{s}{s+1}$ is the positive fixed point of T_s. It follows that $\varprojlim([c_2, c_1], T_s)$ is homeomorphic with two copies of $\varprojlim([p, c_1], T_s^2)$ joined at the endpoint $(\ldots, p, p, p)$. Direct calculation shows that, if L is the orientation preserving affine homeomorphism from $[p, c_1]$ onto $[0, c_1(s^2)]$, then $L \circ T_s^2 \circ L^{-1} = T_{s^2}$ on $[0, c_1(s^2)]$ and hence $\varprojlim([p, c_1], T_s^2)$ is homeomorphic with $\varprojlim([0, 1], T_{s^2})$. $\square$

Lemma 2.2. Suppose that $2^{1/2^n} < s \leq 2^{1/2^{n-1}}$ and $2^{1/2^{n'}} < s' \leq 2^{1/2^{n'-1}}$, $n, n' \in \mathbb{N}$, and suppose that $\varprojlim([0, 1], T_s)$ is homeomorphic with $\varprojlim([0, 1], T_{s'})$. Then $n = n'$ and assuming that the Ingram Conjecture holds for slopes $> \sqrt{2}$, then also $\varprojlim([0, 1], T_{s^{2n-1}})$ is homeomorphic with $\varprojlim([0, 1], T_{(s')^{2n-1}})$.

Proof. For $2^{1/2} < s < 2$, $\varprojlim([0,1], T_s)$ consists of a ray $\mathfrak{C}$ winding onto an indecomposable continuum, namely $\varprojlim([c_2, c_1], T_s)$. It follows from Lemma 2.1 that for $2^{1/2^n} < s < 2^{1/2^{n-1}}$, $\varprojlim([0,1], T_s)$ consists of a ray winding onto a pair of rays, each winding onto a pair of rays,..., each winding onto a pair of rays, each of which winds onto an indecomposable continuum. There are 2^{n-1} of these indecomposable continua, each homeomorphic with the core of the inverse limit space $\varprojlim([0,1], T_{s^{2^{n-1}}})$. Hence if $\varprojlim([0,1], T_s)$ is homeomorphic with $\varprojlim([0,1], T_{s'})$, then $n = n'$ and $\varprojlim([0,1], T_{s^{2^{n-1}}})$ is homeomorphic with $\varprojlim([0,1], T_{(s')^{2^{n-1}}})$. To cover the remaining cases, note that if $s = 2^{1/2^{n-1}}$, then the only alteration needed in the above description of $\varprojlim([0,1], T_s)$ is that at the penultimate level, instead of a pair of rays winding onto a pair of indecomposable subcontinua, we just have two indecomposable subcontinua (each homeomorphic with $\varprojlim([0,1], T_2)$) joined at their common endpoint. It is clear in this case that if $\varprojlim([0,1], T_{s'})$ is homeomorphic with $\varprojlim([0,1], T_s)$, then $s' = s$. $\square$

Definition 2.3. The *arc-length* or $\bar{d}$ metric on $\mathfrak{C}$ is defined as

$$\bar{d}(x,y) = s^p |x_{-p} - y_{-p}|$$

for each p so that $\pi_p : [x, y] \to [0, 1]$ is injective.

If $x, y \in \mathfrak{C}$, then we denote by $[x, y]$ the arc between x and y, and by (x, y) the interior of the arc $[x, y]$. We write $x \preceq y$ if $x \in [\alpha, y]$, *i.e.*, $\bar{d}(\alpha, x) \leq \bar{d}(\alpha, y)$.

Definition 2.4. A continuum is *chainable* if for every $\varepsilon > 0$, there is a cover $\{\ell^1, \ldots, \ell^n\}$ of open sets (called *links*) of diameter $< \varepsilon$ such that $\ell^i \cap \ell^j \neq \emptyset$ if and only if $|i - j| \leq 1$. Such a cover is called a *chain*. Clearly the interval $[0, s/2]$ is chainable. We call $\mathcal{C}_p$ a *natural* chain of $\varprojlim([0,1], T_s)$ if

1. there is a chain $\{I_p^1, I_p^2, \ldots, I_p^n\}$ of $[0, s/2]$, with the relatively open interval I_p^j and I_p^{j+1} adjacent for all $1 \leq j < n - 1$, such that $\ell_p^j := \pi_p^{-1}(I_p^j)$ are the links of $\mathcal{C}_p$;

2. each point $x \in \cup_{i=0}^p T_s^{-i}(c)$ is the boundary point of some link I_p^j;

3. for each i there is j such that $T_s(I_{p+1}^i) \subset I_p^j$.

Let us define width$(\mathcal{C}_p) := \max_j |I_p^j|$. If width$(\mathcal{C}_p) < \varepsilon s^{-p}/2$ then mesh$(\mathcal{C}_p) := \max\{\text{diam}(\ell) : \ell \in \mathcal{C}_p\} < \varepsilon$, which shows that $\varprojlim([0,1], T_s)$ is indeed chainable.

Condition 3. ensures that $\mathcal{C}_{p+1}$ *refines* $\mathcal{C}_p$ (written $\mathcal{C}_{p+1} \preceq \mathcal{C}_p$).

Definition 2.5. A point $x = (\dots, x_{-2}, x_{-1}, x_0) \in \mathfrak{C}$ is called a *p-point* if $x_{-j} = c$ for some $j \geq p$. For the largest such j, the number $L_p(x) := j - p$ is called the *p-level*. In particular, $x_0 = T_s^{p+L_p(x)}(c)$. The ordered set of all p-points of composant $\mathfrak{C}$ is denoted by E_p, and the ordered set of all p-points of p-level l by $E_{p,l}$. Given an arc $A \subset \mathfrak{C}$ with successive p-points $x^0, \dots, x^n$, the *p-folding pattern* of A, denoted by $FP_p(A)$, is the sequence $FP_p(A) = L_p(x^0), \dots, L_p(x^n)$. The *folding pattern of composant* $\mathfrak{C}$, denoted by $FP(\mathfrak{C})$, is the sequence $L_p(z^1), L_p(z^2), \dots, L_p(z^n), \dots$, where $E_p = \{z^1, z^2, \dots, z^n, \dots\}$ and p is any nonnegative integer. Let $q \in \mathbb{N}$, $q > p$, and $E_q = \{y^0, y^1, y^2, \dots\}$. Since σ^{q-p} is an order-preserving homeomorphism of $\mathfrak{C}$, it is easy to see that, for every $i \in \mathbb{N}$, $\sigma^{q-p}(z^i) = y^i$ and $L_p(z^i) = L_q(y^i)$. Therefore, the folding pattern of $\mathfrak{C}$ does not depend on p.

For the above arc A, the projection $\pi_p : A \to [0, s/2]$ need not be injective; so the folding pattern of A can be very long and A may pass through the same link ℓ^j of the natural chain $\mathcal{C}_p$ many times. If A^j is an arc component of $A \cap \ell^j$, then we say that A^j *goes straight* through ℓ^j if $\pi_p|_{A^j}$ is injective; otherwise it *turns* in ℓ^j. If A^j turns in ℓ^j, then A^j contains at least one p-point.

Definition 2.6. Let $\ell^0, \ell^1, \dots, \ell^k$ be those links in $\mathcal{C}_p$ that are successively visited by an arc $A = [u, v] \subset \mathfrak{C}$ (hence $\ell^i \neq \ell^{i+1}$, $\ell^i \cap \ell^{i+1} \neq \emptyset$ and $\ell^i = \ell^{i+2}$ is possible if A turns in ℓ^{i+1}). Let $A^i \subset \ell^i$ be the corresponding arc components such that $\mathrm{Cl}\, A^i$ are subarcs of A. We call the arc A

- *p-link-symmetric* if $\ell^i = \ell^{k-i}$ for $i = 0, \dots, k$;

- *maximal p-link-symmetric* if it is p-link-symmetric and there is no p-link-symmetric arc $B \supset A$ and passing through more links than A;

- *p-symmetric* if $\pi_p(u) = \pi_p(v)$ and if for $A \cap E_p = \{x^0, \dots, x^n\}$ we have $L_p(x^i) = L_p(x^{n-i})$ for every $i = 0, \dots, n$.

In any of these cases, the p-point of $A^{k/2}$ with the highest p-level is called the *center* of A, and the link $\ell^{k/2}$ is called the *central link* of A.

It is easy to see that if A is p-symmetric, then n is even and $L_p(x^{n/2}) = \max\{L_p(x^i) : x^i \in A \cap E_p\}$. Clearly, every p-symmetric arc is p-link-symmetric as well, but the converse does not hold.

Definition 2.7. Let $(s_i)_{i \in \mathbb{N}}$ be a sequence of p-points such that $0 \leq L_p(x) < L_p(s_i)$ for every p-point $x \in (\alpha, s_i)$. We call p-points satisfying this property *snappy*.

Since for every slope $s > 1$ and $p \in \mathbb{N}_0$, the sequence $FP(\mathfrak{C})$ starts as $0\ 1\ 0\ 2\ 0\ 1\ \dots$, and since by definition $L_p(s_1) > 0$, we have $L_p(s_1) = 1$. Also, since $s_i = \sigma^{i-1}(s_1)$, $L_p(s_i) = i$, for every $i \in \mathbb{N}$. Note that the snappy p-points depend on p: if $p \geq q$, then the snappy p-point s_i equals the snappy q-point s_{i+p-q}.

Let us extend the notion of folding pattern as follows. A sequence $e_1, \dots, e_k$ is the *folding pattern* of $T^j|_H$ for an interval $H \subset [0,1]$ if $c_{e_1} = T^j(x_1), \dots, c_{e_k} = T^j(x_k)$, where $x_1 < \cdots < x_k$ are the critical points of T^j on H. (If $0 \in H$, then the folding pattern starts with $*$ by convention, just as $*$ denotes the conventional p-level of α.) In this extended terminology, the p-folding pattern of $[\alpha, s_{j+1}]$ is the same as the folding pattern of T^j on $[0, c_1]$, independently of p.

Measured in arc-length, $\bar{d}(\alpha, s_1) = \frac{1}{2}s^p$, and since $\sigma(s_i) = s_{i+1}$ we obtain

$$\bar{d}(\alpha, s_i) = \frac{1}{2}s^{p-1}s^i \quad \text{for all } i \geq 1. \tag{2.1}$$

3 Maximal Link-Symmetric Arcs

In this section we establish upper bounds for the lengths of p-link-symmetric arcs. The Ingram Conjecture was previously proved for all tent-maps with a (pre)periodic critical point, see [23]. So let as assume from now on that the slope s is such that c is not (pre)periodic. Throughout this section we use the notation $T := T_s$, $a_k := T^k(a)$ for any point or interval (except for the precritical points z_k in Definition 3.3 below), and $\hat{a} := 1 - a$ is the symmetric point around c.

Definition 3.1. Given $\varepsilon > 0$ and $H := [a, b] \subset [0, c_1]$, we say that $T^n|_H$ is *ε-symmetric*, if $|T^n(a+t) - T^n(b-t)| < \varepsilon$ for all $0 \leq t \leq b - a$.

If $\text{width}(\mathcal{C}_p) < \varepsilon$ and the arc $J \subset [\alpha, s_k]$ is p-link-symmetric, then $\pi_{p+k} : J \to H := \pi_{p+k}(J)$ is one-to-one and $T^k|_H$ is ε-symmetric.

Definition 3.2. We say that $T^n|_H$ is *ε-periodic* of period 2η if $|T^n(t) - T^n(t + 2\eta)| < \varepsilon$ for all $t, t + 2\eta \in H$.

If $T^n|_H$ is ε-symmetric around two centers that are η apart, then $T^n|_H$ is ε-periodic with period 2η. We will explain this fact in more detail in the proof of Proposition 3.6, where it is used several times.

Definition 3.3. We call z_k a *closest precritical point* if $T^k(z_k) = c$ and T^k maps $[c, z_k]$ monotonically onto $[c_k, c]$. Clearly, if z_k is a closest precritical points, so is $\hat{z}_k$.

Lemma 3.4. There are infinitely many N and closest precritical points z_N such that
$$\theta_N := \min\{|c_i - c| : 0 < i \le N\} > |z_N - c|.$$

Proof. If c is not recurrent, then $\theta_n \not\to 0$ and the lemma is trivial. So let us assume that c is recurrent, but obviously not periodic. Let n be such that $|c_n - c| = \theta_n$.

If $x \mapsto |T^n(x) - c|$ has a local maximum at c, then $T^n([c, c_n]) \ni c$. Indeed, if this were not the case, then by the choice of n, T^n maps $[c, c_n]$ in a monotone fashion into $[c, c_n]$, which is clearly impossible for tent maps with slope > 1. So in this case, $z_n \in [c_n, \hat{c}_n]$ and the lemma holds with $N = n$.

So assume now that $x \mapsto |T^n(x) - c|$ has a local minimum at c. Take $m \in \mathbb{N}$ minimal such that the closest precritical $z_m \in [\hat{c}_n, c_n]$. We will show that $c_j \notin [\hat{z}_m, z_m]$ for $n < j \le m$. If $j = m$, then $x \mapsto |T^j(x) - c|$ has a local maximum at c, and we can argue as above. So assume by contradiction that $c_j \in [\hat{z}_m, z_m]$ for some $n < j < m$. If $x \mapsto |T^j(x) - c|$ has a local maximum at c, then the closest precritical point z_j satisfies $T^j([c, z_m]) \subset T^j([c, z_j]) = [c_j, c] \subset [\hat{z}_m, c]$ or $[c, z_m]$. This implies that either $[c, z_m]$ or $[\hat{z}_m, c]$ is mapped monotonically into itself by T^j, which is impossible. The remaining possibility is that $x \mapsto |T^j(x) - c|$ has a local minimum at c. In this case, T^{j-n} maps $[z_m, c_n]$ monotonically onto $[w, c_j]$. If $c \in (w, c_j)$, then $m \in \mathbb{N}$ cannot be minimal such that $T^m([c, c_n]) \ni c$. If $c \notin (w, c_j)$, then $w \in [\hat{c}_n, c_n] \cap T^{(j-n)-m}(c)$, and since $-m < (j - n) - m < 0$, m is again not minimal such that $T^m([c, c_n]) \ni c$.

Take $N = m$ and the lemma follows. $\qquad\square$

Take N_0 as in Lemma 3.4 and so large that $s^{N_0} > 100$. Let $N \ge N_0$ from Lemma 3.4 be so large that
$$\delta := |z_N - c| < |z_{N_0} - c|/100. \tag{3.1}$$
Then $|c_n - c| \ge s^n|z_n - c| \ge s^{N_0}|z_N - c| > 100\delta$ for every $N_0 \le n \le N$ by the choice of N_0 and $|c_n - c| > |z_{N_0} - c| > 100\delta$ for $n \le N_0$ by the choice of N.

Lemma 3.5. Given δ as in (3.1), there exists $r_0 = r_0(\delta)$ such that for every interval $\tilde{J}$ with $|\tilde{J}| \ge 22\delta$, there exist $l \le r_0 N$ and an interval J with $|J| \ge 18\delta$ and concentric with $\tilde{J}$, such that $T_s^l|_J$ is monotone and $J_l := T^l(J) \supset [c - \delta, c + \delta]$.

Proof. Let x be the center of $\tilde{J}$ and take $m \ge 0$ minimal such that $\tilde{J}_m \ni c$; hence $T^m|_{\tilde{J}}$ is monotone.

Clearly, $m \le (r_0 - 1)N$ for some $r_0 \ge 1$ depending only on δ. If $\partial \tilde{J}_m$ is δ-close to c, then we take $J' \subset \tilde{J}$ centered at x and slightly smaller such that $c \in \partial J'_m$ and $m' > m$

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minimal such that $J'_{m'}$ contains c in its interior. Since $|J'_m| > 20\delta$, it contains z_N or $\hat{z}_N$ as in (3.1), and $m' - m \leq N$ and $|c_{m'-m} - c| \geq \delta$ by Lemma 3.4.

If at iterate m' the other boundary point of J' is δ-close to c, then $m' - m < N$. We take the interval $J'' \subset J'$ centered at x slightly smaller such that $c \in T^{m'}(\partial J'')$ and take $m'' > m'$ minimal such that c is an interior point of $T^{m''}(J'')$. Since $T^{m'}(z_N) \in T^{m'}(J'')$, and by (3.1) again, $m \leq m' \leq m'' \leq m + N$ and $\partial J''_{m''}$ is not δ-close to c. In each case, there is $l \leq r_0 N$ and $J \in \{\tilde{J}, J', J''\}$ so that the lemma holds. $\qquad\square$

For interval $H =: [a, b]$ with center x we formulate the following property:

$$c \in H \text{ and } \delta < \min\{|c - a|, |c - b|, |c - x|\}. \tag{3.2}$$

Proposition 3.6. Assume that $s \in [1, 2]$ is such that c is not (pre)periodic. There exists $\varepsilon > 0$ such that if H satisfies (3.2), then $T^n|_H$ is not ε-symmetric for any $n \in \mathbb{N}_0$.

Proof. We will prove Proposition 3.6 using the induction hypothesis:

$$\text{if } H \text{ satisfies (3.2), then } T^n|_H \text{ is not } \varepsilon\text{-symmetric.} \tag{IH_n}$$

Take N_0, N and δ as in (3.1), r_0 as in Lemma 3.5 and H that satisfies (3.2).

Let $\varepsilon \in (0, \delta)$ be so small that

$$\varepsilon < \min\{|c_i - c_j| : 0 \leq i < j \leq (2 + r_0)N\}. \tag{3.3}$$

Since c lies off-center in H by at least δ, by the choice of ε, (IH_k) holds for all $k \leq (2 + r_0)N$. Assume now that (IH_j) holds for all $j < n$. We will prove (IH_n), but first, continuing with the interval $\tilde{J}$ of Lemma 3.5, we prove the following lemma.

Lemma 3.7. Let $\tilde{J}$ be an interval of length $|\tilde{J}| \geq 22\delta$ centered at c_k for some $1 \leq k \leq 2N$. If $T^j|_{\tilde{J}}$ is ε-symmetric for some $0 \leq j \leq n$, then the interval $J_l := T^l(J)$ from Lemma 3.5 satisfies condition (3.2).

Proof. We know already from Lemma 3.5 that $J_l \supset [c - \delta, c + \delta]$. Hence if (3.2) fails, then $\eta := |c_{k+l} - c| \leq \delta$. Since $T^l|_J$ is monotone, $j > l$. Therefore $T^{j-l}|_{J_l}$ is ε-symmetric around c_{k+l} and symmetric around c, and it follows that $T^{j-l}|_{J_l}$ is ε-periodic with period 2η. Indeed, by symmetry around c, $T^{j-l}_{J_l}$ is ε-symmetric around the symmetric point $\hat{c}_{k+l}$. Hence $T^{j-l}_{J_l}$ must also be ε-symmetric around the points $c \pm 2\eta$, which are the reflections of c in c_{k+l} and $\hat{c}_{k+l}$, etc. Extending these symmetries, we see that $|T^{j-l}(t) - T^{j-l}(t + 2\eta)| < \varepsilon$ for all $t, t + 2\eta \in J_l$, so $T^{j-l}|_{J_l}$ is ε-periodic with period 2η. Even more, $T^{j-l}|_{J_l}$ is ε-symmetric around $c + 2i\eta$ on every separate subarc $P_i := [c + (2i - 1)\eta, c + (2i + 1)\eta] \subset J_l$.

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Recall that $1 \le k \le 2N$ and $l \le r_0 N$, so we have $\eta > \varepsilon$ by the choice of ε in (3.3). Since $|J_l| \ge 18\delta = 18|z_N - c|$, one of the components of $J_l \setminus \{c\}$, say the one containing $z_N > c$, has length $\ge 9\delta$. We can take $r \le N$ minimal such that $z_r \in [c + \delta, c + 8.9\delta]$. Take $i \in \mathbb{Z}$ such that if

$$z_r \in \begin{cases} (c, c + 4.3\delta], & \text{then } c + 2i\eta \in (z_r + 0.1\delta, z_r + 2.1\delta), \\ (c + 4.3\delta, c + 8.9\delta], & \text{then } c + 2i\eta \in (z_r - 2.1\delta, z_r - 0.1\delta). \end{cases} \tag{3.4}$$

Let $H \subset J_l$ be the longest interval centered at $x := c + 2i\eta$ on which $T^r|_H$ is monotone. Then $H \ni z_r$, and $T^{j-l}|_H$ and $T^{j-l-r}|_{H_r}$ are ε-symmetric. We will show that H_r satisfies (3.2). Indeed, since $|z_r - c| \le 9\delta < |z_{N_0} - c|/10$ (so $r > N_0$) by (3.1) and $|x - z_r| \ge \delta/10$, we have $|x_r - c| = s^r|x - z_r| \ge 2^{N_0/2}\delta/10 > \delta$. If $|z_r - \partial H| \ge \delta/10$, then $|c - \partial H_r| > \delta$ for the same reason. If on the other hand there is a point $y \in \partial H$ such that $|y - z_r| < \delta/10$, then y has to be a precritical point. By the choice of r, $y = z_{r'} \in (c + 8.9\delta, c + 9\delta]$ for some $r' < r$. By the choice of N and Lemma 3.4, $|y_r - c| = |c_{r-r'} - c| \ge \delta$.

This shows that H_r satisfies (3.2), but also $T^{j-l-r}|_{H_r}$ is ε-symmetric around x_r, and this contradicts (IH_{j-l-r}), proving this lemma. $\qquad\square$

Combining the induction hypothesis (IH_n) and Lemma 3.7, we have proved the following stronger property.

Corollary 3.8. If $\tilde{J}$ is centered at c_k for some $1 \le k \le 2N$ and $|\tilde{J}| \ge 22\delta$, then $T^j|_{\tilde{J}}$ is not ε-symmetric for $j \le n$.

Now we continue the induction on n and assume by contradiction that $T^n|_H$ is ε-symmetric for some H satisfying (3.2) and for ε satisfying (3.3). Let $[a', b'] := H' \subset H$ be centered around x such that $c \in \partial H'$. Assume without loss of generality that $c = a'$ is the left endpoint of H', and let L and R be intervals of length δ at the left and right side adjacent to H'. Since $|H'| \ge \delta$, so $H' \ni z_N$ or $\hat{z}_N$, there is $0 < k \le N$ minimal such that $c \in H'_k$. Clearly $|H'_k| > |L_k| = |R_k| \ge 100\delta$. We distinguish four cases:

Case I: H'_k satisfies (3.2). Then by (IH_{n-k}), $T^{n-k}|_{H'_k}$ cannot be ε-symmetric, and neither can $T^n|_{H'}$ or $T^n|_H$.

Case II: $|x_k - c| < \delta$, see Figure 1 (left). If the length of the interval $T^{n-k}([x_k, c])$ exceeds ε, then since T^{n-k} is also symmetric around c, T^{n-k} must be ε-symmetric on H'_k both with center x_k and with center $\hat{x}_k$, and therefore ε-periodic on H' with period $2\eta := 2|x_k - c|$. We use the same argument as in the proof of Lemma 3.7: T^{n-k} is ε-symmetric on each interval $P_i := [c + (2i - 1)\eta, c + (2i + 1)\eta]$ for each $i \in \mathbb{Z}$ such that $P_i \subset H'_k$. Since $|H'_k| \ge 100\delta \ge 100\eta$, $P_i \subset H'_k$ for at least $-25 \le i \le 25$. Take $r \le N$ minimal such that
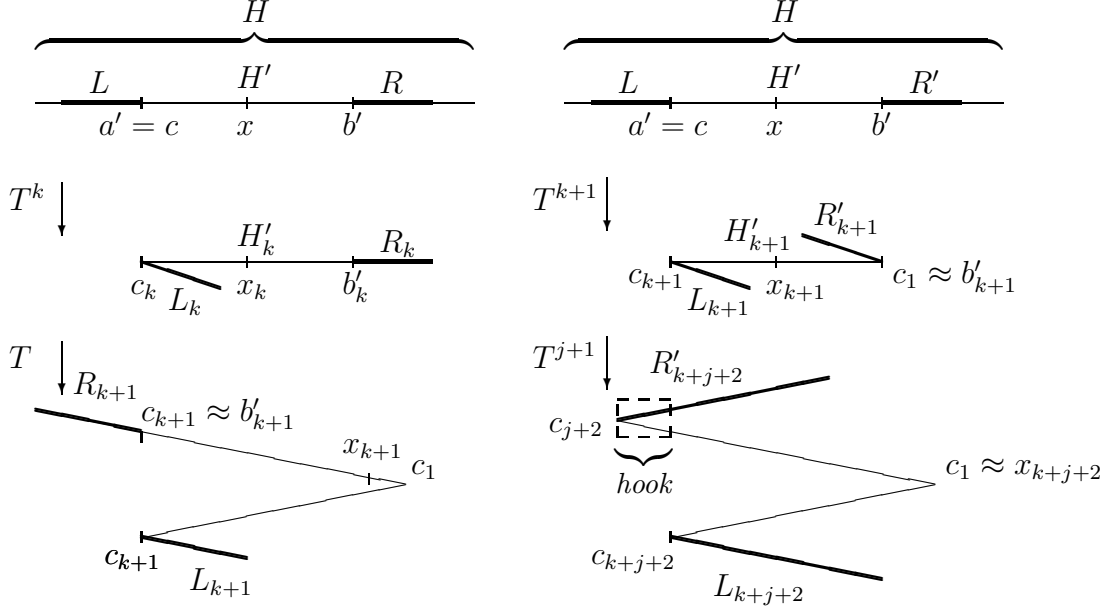
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Figure 1: An illustration of Cases II (left) and IV (right).

$[z_r - \delta/10, z_r + \delta/10] \subset H'_k$, and $i \in \mathbb{Z}$ as in (3.4), and $H'' \subset H'$ such that H''_k is the maximal interval centered at $c + 2i\eta$ on which T^r is monotone. As before, $T^{n-(k+r)}|_{H''_{k+r}}$ is the ε-symmetric but H''_{k+r} satisfies (3.2). This would contradict $(\mathrm{IH}_{n-(k+r)})$, so it cannot occur.

If on the other hand the length of $T^{n-k}([x_k, c])$ is less then ε, then we might as well have chosen x such that $x_k = c$. This means that the intervals L_{k+1} and R_{k+1} are adjacent, see Figure 1 (bottom left). More precisely, they are adjacent except for an error which does not show at ε-scale under the iterate $T^{n-(k+1)}$, so by a negligible adjustment, we can assume that they form an interval of length $\geq 100\delta$ with center c_{k+1}. Since $k + 1 \leq 2N$, Corollary 3.8 implies that $T^{n-(k+1)}|_{L_{k+1} \cup R_{k+1}}$ and hence $T^n|_H$ are not ε-symmetric.

Case III: $|a'_k - c| < \delta$. Since $k \leq N$, the choice of δ renders this impossible.

Case IV: $|b'_k - c| < \delta$, see Figure 1 (right). Replace R by the largest interval $R' \subset H \cup R$ with $R' \cap R \neq \emptyset$ such that $c \in \partial R'_k$ and $T^k|_{R'}$ is monotone. If $c \in \partial R'_l$ for some $0 \leq l < k$, then $R'_k = [c, c_{k-l}]$, so $|R'_k| \geq \delta$ by Lemma 3.4. Also rename $H' \setminus R'$ to H'. Hence $T^{k+1}|_{L \cup H' \cup R'}$ has three branches, $s\delta \leq |R'_{k+1}|$ and $100\delta \leq |L_{k+1}| \leq |H'_{k+1}|$.

Let $j > 0$ be minimal such that $T^{k+j+1}(H') \ni c$. If $H'_{k+j+1} = [c_{k+j+1}, c_{j+1}]$, which is centered at x_{k+j+1}, satisfies (3.2), then we can invoke $(\mathrm{IH}_{n-(k+j+1)})$, so assume that this is not the case. Since $|L| \geq \delta$, so $L \ni z_N$ or $\hat{z}_N$, we have $j \leq k + j + 1 \leq N$. Therefore both $|c_{j+1} - c| > \delta$ and $|c_{k+j+1} - c| \geq \delta$.

Thus if (3.2) fails, we must have $|x_{k+j+1} - c| < \delta$. If in the remaining $n - (k + j + 1)$ iterates, the arc $[x_{k+j+1}, c]$ grows to length $> \varepsilon$, then, as in Case II, $T^n|_{H'}$ must contain a large ε-periodic arc, to which we apply the same argument as in Case II (*i.e.*, the argument of Lemma 3.7). The remaining possibility is that x_{k+j+1} is so close to c that on an ε-scale, we may as well assume that $x_{n+k+1} = c$.

Both $c_{k+j+2} = a'_{k+j+2}$ and $c_{j+2} \approx b'_{k+j+2}$ are local minima of $T^{k+j+2}|_{L \cup H' \cup R'}$, see Figure 1 (bottom right). Assume without loss of generality that $c_{j+2} < c_{k+j+2}$, so R'_{k+j+2} has a small extra hook before joining up with L_{k+j+2}. As we assumed that $T^n|_H$ is ε-symmetric, the effect of this hook needs to be 'ε-repeated' near a' in L. But L_{k+j+2} and R'_{k+j+2} overlap, so in R', the same effect needs to be ε-repeated next to the first hook. Continuing this way, we find that $T^{n-(k+j+2)}$ is ε-periodic over the entire length of R'_{k+j+2}.

Take i minimal such that $R'' := T^i(R'_{k+j+2}) \ni c$. Since $|R'_k| \geq \delta$ we have $j+i+2 < N$, $|R''| \geq 100\delta$ and $|\partial R'' - c| \geq \delta$. Therefore $T^{n-(k+j+i+2)}|_{R''}$ is ε-periodic of period 2η, where the length of the hook after i more iterates is $\eta := |c_{j+i+2} - c_{k+j+i+2}| > \varepsilon$, because $k + j + i + 2 \leq 2N$ and by the choice of ε in (3.3). If $\eta < 10\delta < |R''|/10$, then $T^{n-(k+j+i+2)}|_{R''}$ is ε-periodic with at least 5 adjacent intervals P of length 2η around the center of which $T^{n-(k+j+i+2)}|_{R''}$ is ε-symmetric. So we can find a new interval $H'' \subset R''$ centered around the center of one of these Ps such that H'' satisfies (3.2). But this contradicts $(\mathrm{IH}_{n-(k+j+i+2)})$.

If $\eta \geq 10\delta$, then we let H'' be the arc of length 22δ centered at $c_{k+j+i+2}$. Again, since $k + j + i + 2 \leq 2N$, the iterate $T^{n-(k+j+i+2)}$ cannot be ε-symmetric on H'' by Corollary 3.8. But then the assumed ε-symmetry of $T^n|_H$ does not extend beyond H', and Case IV follows.

This proves the inductive step and hence the proposition. $\qquad\square$

Let $\kappa := \min\{i \geq 3 : c_i \geq c\}$. Then $\kappa < \infty$ provided $1 < s < 2$. Let $\cdots < c_{-3} < c_{-2} < c_{-1} < c_0 = c$ be the successive precritical points on the left of c with $T^j(c_{-j}) = c$. Since $c_{\kappa-1} < c < c_\kappa$, we have $c_{2-\kappa} < c_2 < c_{3-\kappa}$. Let $\delta = |z_N - c|$ as in (3.1) be so small (*i.e.*, N as in Lemma 3.4 so large) that

$$\delta < \frac{1}{30} \min\{|c_{-1} - c_{-2}|, |c_{-1} - \hat{c}_1|, |c_2 - c_{2-\kappa}|\}, \tag{3.5}$$

where $\hat{c}_1 = 1 - c_1 = 1 - s/2$. Assume that $s \in [1, 2]$ is such that c is not (pre)periodic, and take ε is as in (3.3) in the proof of Proposition 3.6.

Let $(A_i)_{i \in \mathbb{N}}$ be the sequence of maximal p-link-symmetric arcs with center s_i for every $i \in \mathbb{N}$. Recall that $(s_i)_{i \in \mathbb{N}}$ is the sequence of snappy p-points (see Definition 2.7) and that

$\mathrm{width}(\mathcal{C}_p) := \max_j |I_p^j|$.

Lemma 3.9. If $\mathrm{width}(\mathcal{C}_p) < \varepsilon$, then A_i contains exactly κ snappy p-points for each $i \geq \kappa - 1$, namely $s_{i-\kappa+2}, s_{i-\kappa+3}, \ldots, s_{i+1}$, and $s_{i-\kappa+2}$ is an interior point of A_i.

Proof. Let H be the interval centered at c_2 such that c is the left endpoint of $H_{\kappa-2} := T^{\kappa-2}(H)$. Then $|H| \geq 22\delta$ by the choice of δ, so by Proposition 3.6 and Lemma 3.7 in particular, $T^{p+i-1}|_H$ cannot be ε-symmetric.

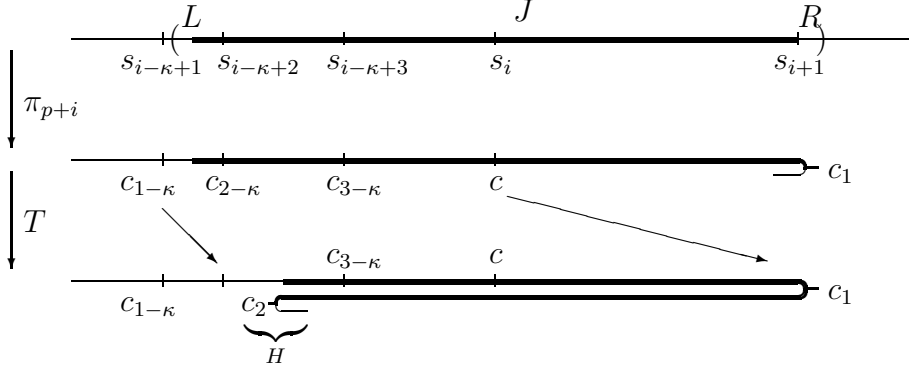

Figure 2: The arc J and its image under π_{p+i} and $T \circ \pi_{p+i} = \pi_{p+i-1}$.

Let $J = [x, s_{i+1}]$ be such that $J \ni s_i$ and $\bar{d}(s_i, s_{i+1}) = \bar{d}(x, s_i)$, where $\bar{d}$ is defined in Definition 2.3. Then π_{p+i-1} maps J in a 2-to-1 fashion onto $[c_2, c_1]$, with $\pi_{p+i-1}(s_i) = c_1$ and $\pi_{p+i-1}(s_{i+1}) = c_2$. Therefore J is p-symmetric and also p-link-symmetric around s_i. Since $c_{2-\kappa} < c_2 < c_{3-\kappa}$, we have $\pi_{p+i-1}(J) \not\ni c_{2-\kappa}$. Extend J on either side by equally long arcs L and R such that $\pi_{p+i-1}(L \cup R) = H$, see Figure 2. Since $T^{p+i-1}|_H$ is not ε-symmetric, $A_i \not\supset L \cup J \cup R$ provided $\mathrm{width}(\mathcal{C}_p) < \varepsilon$. Hence $A_i \not\ni s_{i-\kappa+1}$ as claimed. $\square$

Remark 3.10. The bound κ in this lemma is not sharp if T_s has a periodic critical point. For example, for the tent map with $c_2 < c = c_3 < c_1$, the folding pattern is

$$FP(\mathfrak{C}) = {*}\,0\,\underline{1}\,0\,2\,0\,1\,\underline{3}\,\overbrace{1\,0\,2\,0\,\underline{4}\,0\,2\,0\,1\,3\,1\,0\,\underline{5}\,0\,1\,3\,1\,0\,2\,0\,4\,0\,2\,0\,1}^{\text{maximal } p\text{-symmetric}}\,\underbrace{\underline{6}\,1\,0\,2\,0\,4\,0\,2\,0\,1}_{\text{maximal } p\text{-link-symmetric}}\ldots$$

where p-levels of snappy p-points are underlined and $*$ denotes the conventional p-level of α. Since c has period 3, so $c_a = c_{a+3b}$ for all $a, b \in \mathbb{N}$, p-link-symmetric arcs can be longer than p-symmetric arcs. Indeed, the maximal p-symmetric arc centered at snappy point s_5 stretches from s_3 to s_6, while maximal p-link-symmetric arc centered at s_5 stretches almost from α to some point with p-level 2. This property holds for all snappy points: the maximal p-link-symmetric arc around s_i contains s_j for all $j \leq i + 1$.

A preperiodic example is $s = 2$, *i.e.*, $\varprojlim([0, 1], T_s)$ is the Knaster continuum.

Lemma 3.11. Assume that $\text{width}(\mathcal{C}_p) < \varepsilon$ and fix $i \in \mathbb{N}$, $i > \kappa - 1$, and let ℓ^i and ℓ^{i-1} be the links of $\mathcal{C}_p$ containing s_i and s_{i-1} respectively. Let y be such that $s_{i-1} \prec y \prec s_i$ and y is not contained in the same arc-component of ℓ^i as s_i, nor in the same arc-component of ℓ^{i-1} as s_{i-1}. Then the maximal p-link-symmetric arc J with center y contains at most one snappy p-point, and $J \subset A_i$.

Proof. Let $\cdots < c_{-2} < c_{-1} < c_0 = c$ be the successive precritical points to the left of c with $T^j(c_{-j}) = c$. Since A_i contains s_{i+1} and its symmetric point around s_i (at least as boundary points), we have $\pi_{p+i}(A_i) \supset [\hat{c}_1, c_1] \supset [c_{-1}, c]$. Let $H := \pi_{p+i}(J)$ with center $x := \pi_{p+i}(y) \in [c_{-1}, c]$. Assume by contradiction that J contains two snappy p-points, or that $J \not\subset A_i$. Then $|H| \geq 22\delta$ by the choice of δ in (3.5).

Let $w := (c_{-1} + c)/2$. We distinguish four cases.

1. $c - \delta < x < c$. If $|T^i([x, c])| \leq \varepsilon$, then we cannot '$\varepsilon$-distinguish' x from c, violating our assumption that y and s_i do not belong to the same arc-component of the same link. If $|T^i([x, c])| > \varepsilon$, then T^i is ε-symmetric on H with centers x and c, so T^i is ε-periodic on H with period $2|x - c|$. This leads to a contradiction by the argument of the proof of Lemma 3.7.

2. $w \leq x \leq c - \delta$. Then H satisfies (3.2), so by Proposition 3.6, $T^i|_H$ cannot be ε-symmetric.

3. $c_{-1} + \delta/s \leq x < w$. Then by assumption H contains one of c, c_{-2} or $\hat{c}_1$ (whence $|H| \geq 22\delta$), and hence $T(H \cap [c_{-2}, c] \cap [\hat{c}_1, c])$ satisfies (3.2), so $T^i|_H$ cannot be ε-symmetric by Proposition 3.6.

4. $c_{-1} < x < c_{-1} + \delta/s$. If $|T^i([c_{-1}, x])| \leq \varepsilon$, then we cannot '$\varepsilon$-distinguish' x from c, violating the assumption that y and s_{i-1} are not contained in the same arc component of ℓ^{i-1}. If $|T^i([c_{-1}, x])| > \varepsilon$ and again, H by assumption contains one of c, c_{-2} or $\hat{c}_1$ (so $|H| \geq 22\delta$), then T^{i-1} is ε-periodic on $T(H)$ which again leads to a contradiction by the argument of the proof of Lemma 3.7.

This proves the lemma. $\qquad\square$

4 Link-Symmetric Arcs and Homeomorphisms

In this section we study the action of homeomorphisms $h : \varprojlim([0, 1], T_{s'}) \to \varprojlim([0, 1], T_s)$ on snappy q-points and q-points in general. Let $q, p, g \in \mathbb{N}_0$ be such that

$$h(\mathcal{C}_q) \preceq \mathcal{C}_p \preceq h(\mathcal{C}_g).$$

Recall that we assumed the slopes s' and s to be such that the critical points c' and c are not (pre)periodic. Clearly h maps the zero-composant $\mathfrak{C}'$ of $\varprojlim([0,1], T_{s'})$ to the zero-composant $\mathfrak{C}$ of $\varprojlim([0,1], T_s)$, and in particular the endpoint α' of $\mathfrak{C}'$ to the endpoint α of $\mathfrak{C}$. Let $\kappa' := \min\{i \geq 3 : c_i' \geq c'\}$, where $c_i' = T_{s'}^i(c')$. Let us denote the snappy q-points (i.e., associated with $\mathcal{C}_q$) by s_i' and the snappy g-points by s_i''. Therefore, snappy q-point s_i' is the same as snappy g-point s_{i+q-g}''. Similarly, let A_i' be the maximal q-link-symmetric arc centered at s_i' while as before, A_i denotes the maximal p-link-symmetric arc centered at s_i

Since A_i' is q-link-symmetric, and $h(\mathcal{C}_q) \preceq \mathcal{C}_p$, the image $D_i := h(A_i') \subset \mathfrak{C}$ is p-link-symmetric and therefore has a well-defined center, we denote it as m_i, and a well-defined central link ℓ_p (see Definition 2.6). In fact, $h(s_i')$ and m_i belong to the central link ℓ_p and m_i is the p-point with the highest p-level of all p-points of the arc component of ℓ_p which contains $h(s_i)$. Let $M_i := L_p(m_i)$.

Theorem 4.1. $M_{i+1} = M_i + 1$ for all sufficiently large integers $i \in \mathbb{N}$.

Proof. Without loss of generality we can assume that $s' \geq s$, so that $\kappa' \geq \kappa$. We prove first that if $N \geq \kappa$ is so large that m_N lies beyond the κ-th snappy p-point of $\mathfrak{C}$, then $L_p(y) < M_N$, for every $y \in (\alpha, m_N)$; i.e., m_N is snappy.

Assume by contradiction that there exists $y \in (\alpha, m_N)$ such that $L_p(y) \geq M_N$. By taking $L_p(y)$ maximal with this property, we can assume that $y = s_{j-1} \prec m_N \prec s_j$ for some $j > \kappa$. More precisely, m_N is not contained in the same arc-component of the link containing s_{j-1} as s_{j-1}, and similarly for s_j. Lemma 3.11 implies that D_N contains at most one snappy p-point and that $D_N \subset A_j$. Let us denote by B the p-link-symmetric arc such that s_j is the center of B, $D_N \subset B \subset A_j$ and $\partial D_N \cap \partial B \neq \emptyset$ (see Figure 3). Since $\mathcal{C}_p \preceq h(\mathcal{C}_g)$, the arc $B'' = \sigma^{q-g} \circ h^{-1}(B)$ is g-link-symmetric and contains the arc $\sigma^{q-g} \circ h^{-1}(D_N) = \sigma^{q-g}(A_N')$. The center z'' of B'' is the center of the arc component of the central link ℓ_g of B'' containing $\sigma^{q-g} \circ h^{-1}(s_j)$. By Lemma 3.9, A_N' contains κ' snappy q-points $s_{N-\kappa'+2}', \ldots, s_N', s_{N+1}'$.

The map σ^{q-g} maps the κ' snappy q-points $s_i' \in A_N'$ to the κ' snappy g-points $s_{i+q-g}'' \in \sigma^{q-g}(A_N')$, and B'' contains at least these κ' snappy g-points. If the center z'' of B'' is not snappy, then B'' contains at most one snappy g-point by Lemma 3.11, so we have a contradiction. Otherwise, if z'' is snappy, then even if z'' is the right-most snappy g-point of $\sigma^{q-g}(A_N')$, then still B'' contains $\kappa' - 1$ snappy g-points on the left of the center z'', contradicting Lemma 3.9. Therefore, m_N is snappy.

Let us consider the arc $D_{N+\kappa'-2} = h(A_{N+\kappa'-2}')$. Since $L_q(s_{i+1}') - L_q(s_i') = 1$, the
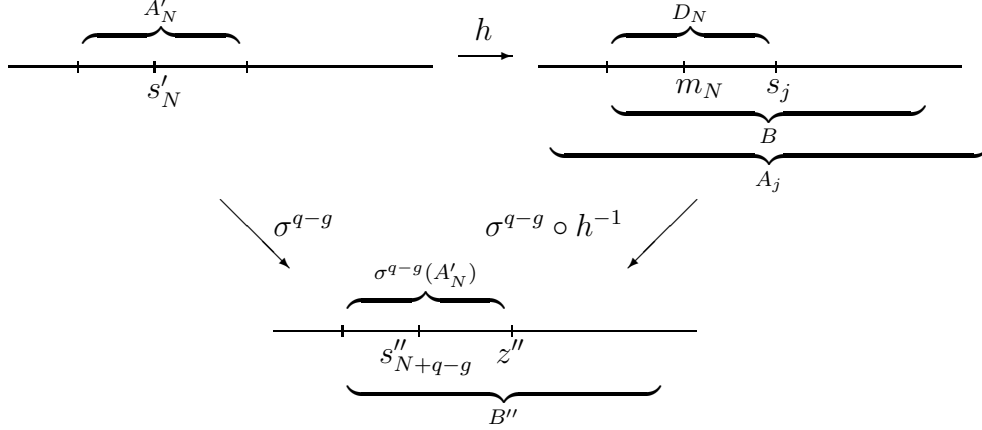
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Figure 3: The relations between relative snappy points and arcs in $\mathcal{C}_q$ (left), $\mathcal{C}_p$ (right), and $\mathcal{C}_g$ (bottom).

arc $[s'_i, s'_{i+1}]$ contains a q-point of every q-level less than i, so contains q-points of q-levels 1 and 2. Therefore, $\pi_q([s'_i, s'_{i+1}]) = [c_2, c_1]$. Note that two different points from $s'_N, \ldots, s'_{N+\kappa'-1} \in A'_{N+\kappa'-2}$ can be mapped into the same link, say ℓ_p of $\mathcal{C}_p$, but cannot be mapped into the same arc component of ℓ_p. Indeed, if $h([s'_i, s'_{i+1}]) \subset A$, where A is a arc component of ℓ_p, then $h(\mathcal{C}_q) \subset \ell_p$, a contradiction. Therefore, $s_{M_N}, \ldots, s_{M_{N+\kappa'-1}}$ are all different.

So, the arc $D_{N+\kappa'-2}$ is p-link-symmetric and contains at least κ' snappy p-points, $s_{M_N}, \ldots, s_{M_{N+\kappa'-1}}$. By Lemma 3.9, the maximal p-link-symmetric arc $A_{M_{N+\kappa'-2}}$ centered at the snappy p-point $s_{M_{N+\kappa'-2}}$ contains κ snappy p-points, $s_{M_{N+\kappa'-2}-\kappa+2}, \ldots, s_{M_{N+\kappa'-2}}, s_{M_{N+\kappa'-2}+1}$. Therefore, $D_{N+\kappa'-2} \subseteq A_{M_{N+\kappa'-2}}$, $\kappa' = \kappa$, $s_{M_{N+i}} = s_{M_N+i}$ and $M_{N+i} = M_N + i$ for all $0 \le i \le \kappa - 1$. By induction we get $M_{N+i} = M_N + i$ for all $i \in \mathbb{N}_0$ as well. $\square$

Every snappy p-point $s_i \in \mathfrak{C}$ can be contained in at most two links of $\mathcal{C}_p$, and one of them is always the central link of A_i, which we will denote by $\ell_p^{s_i}$. Let K_{s_i} be the arc component of $\ell_p^{s_i}$ containing s_i. Given a p-point $x \in \mathfrak{C}$ with $L_p(x) = l$, there can be two links of $\mathcal{C}_p$ containing x, but one of them is always $\ell_p^{s_l}$. We denote the arc component of $\ell_p^{s_l}$ containing x by K_x. Let $\ell_q^{s'_i} \in \mathcal{C}_q$ and $K_{s'_i} \subset \ell_q^{s'_i}$ be the similar notation related to $\mathfrak{C}'$ and $\mathcal{C}_q$. Also, for a q-point x' of $\mathfrak{C}'$ with $L_q(x') = k$ let the arc component of $\ell_q^{s'_k}$ containing x' be denoted by $K_{x'}$.

Proposition 4.2. There exists $M \in \mathbb{Z}$ such that the following holds:

(1) Let $l \in \mathbb{N}$ and let x' be a q-point with $L_q(x') = l$. Then $u = h(x') \in \ell_p^{s_{l+M}}$ and the arc component $K_u \subset \ell_p^{s_{l+M}}$ containing u, also contains a p-point x such that

$$L_p(x) = l + M.$$

(2) For $l \in \mathbb{N}_0$ and $i \in \mathbb{N}$, the number of q-points in $[s_i', s_{i+1}']$ with q-level l is the same as the number of p-points in $[s_{M+i}, s_{M+i+1}]$ with p-level $M + l$.

Proof. (1) Recall that the set of q-points in $\mathfrak{C}'$ is denoted by E_q'. By Theorem 4.1, there exists $M \in \mathbb{Z}$ such that $a_i = h(s_i') \in \ell_p^{s_{M+i}}$ for every $i \in \mathbb{N}_0$ and the arc component K_{a_i} of $\ell_p^{s_{M+i}}$ contains s_{M+i}. Therefore, statement (1) is true for all snappy q-points.

Also $h([s_1', s_2']) = [a_1, a_2]$, $s_{M+1} \in K_{a_1}$ and $s_{M+2} \in K_{a_2}$. Let q-point $x_1' \in [s_2', s_3']$ be such that the arc $[s_1', x_1']$ is q-symmetric with center s_2'. Then $h([s_1', x_1'])$ is p-link-symmetric with center s_{M+2}. Since there exists a unique p-point b_1 such that the arc $[s_{M+1}, b_1]$ is p-symmetric with center s_{M+2}, we have $h(x_1') \in K_{b_1}$, see Figure 4. Also $L_q(x_1') = 1$ and $L_p(b_1) = M + 1$.
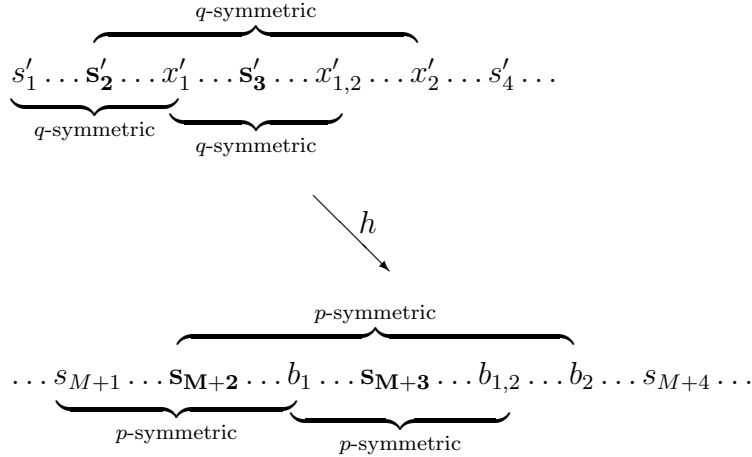

Figure 4: The configuration of symmetric arcs.

We have $h([s_2', s_3']) = [a_2, a_3]$, $s_{M+2} \in K_{a_2}$ and $s_{M+3} \in K_{a_3}$. Let the q-point $x_2' \in [s_3', s_4']$ be such that the arc $[s_2', x_2']$ is q-symmetric with center s_3'. Therefore $h([s_2', x_2'])$ is p-link-symmetric with center s_{M+3}. There exists a unique p-point b_2 such that the arc $[s_{M+2}, b_2]$ is p-symmetric with center s_{M+3}, so $h(x_2') \in K_{b_2}$. Also $L_q(x_2') = 2$ and $L_p(b_2) = M + 2$. Since $[s_2', x_2']$ is q-symmetric, there exists a q-point $x_{1,2}' \in [s_3', x_2']$ such that the arc $[x_1', x_{1,2}']$ is q-symmetric with center s_3'. Then $h([x_1', x_{1,2}'])$ is p-link-symmetric with center s_{M+3}. Since there exists a unique p-point $b_{1,2}$ such that the arc $[b_1, b_{1,2}]$ is p-symmetric with center s_{M+3}, we have $h(x_{1,2}') \in K_{b_{1,2}}$, see Figure 4. Also $L_q(x_{1,2}') = 1$ and $L_p(b_{1,2}) = M + 1$.

The proof of (1) follows by induction. Suppose at step k we have $h([s'_k, s'_{k+1}]) = [a_k, a_{k+1}]$, $s_{M+k} \in K_{a_k}$ and $s_{M+k+1} \in K_{a_{k+1}}$, see Figure 5. Let again q-point $x'_k \in [s'_{k+1}, s'_{k+2}]$ be such that the arc $[s'_k, x'_k]$ is q-symmetric with center s'_{k+1}. Then $h([s'_k, x'_k])$ is p-link-symmetric with center s_{M+k+1}. The unique p-point b_k such that $[s_{M+k}, b_k]$ is p-symmetric with center s_{M+k+1} satisfies $h(x'_k) \in K_{b_k}$. Also $L_q(x'_k) = k$ and $L_p(b_k) = M+k$.
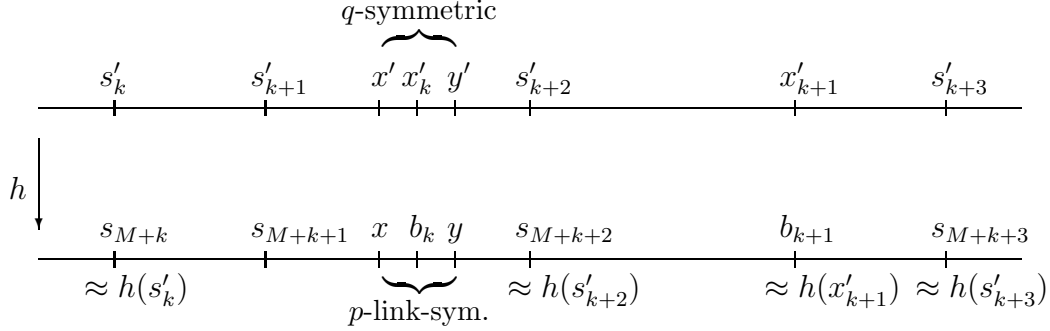

Figure 5: The relative point in the induction step. Here $\approx$ stands for "belongs to the same arc component in the same link".

Let us suppose by induction that for every q-point $x' \in E_q$, $L_q(x') > 0$, $x' \prec x'_k$, we have $u = h(x') \in \ell_p^{s_r+M}$, where $r = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_r+M}$ contains a p-point x such that $L_p(x) = r + M$. Since $L_q(x'_k) = k$, $L_q(s'_{k+1}) = k + 1$ and $L_q(s'_{k+2}) = k + 2$, for every q-point $x' \in (s'_{k+1}, s'_{k+2})$, $x' \neq x'_k$, we have $L_q(x') < L_q(x'_k)$. Hence for every q-point $y' \in (x'_k, s'_{k+2})$ there exists a q-point $x' \in (s'_{k+1}, x'_k)$ such that the arc $[x', y']$ is q-symmetric with center x'_k. So the arc $h([x', y'])$ is p-link-symmetric with center b_k. The induction hypothesis implies that for $u = h(x')$, the arc component $K_u \in \ell_p^{s_r+M}$ contains a p-point x such that $L_p(x) = r + M$, where $r = L_q(x')$.

Since $L_p(b_k) = M+k$, $L_p(s_{M+k+1}) = M+k+1$ and $L_p(s_{M+k+2}) = M+k+2$, we have $L_p(v) < L_p(b_k)$ for every p-point $v \in (s_{M+k+1}, s_{M+k+2})$, $v \neq b_k$. Hence for every p-point $v \in (b_k, s_{M+k+2})$ there exists a p-point $w \in (s_{M+k+1}, b_k)$ such that the arc $[w, v]$ is p-symmetric with center b_k. Therefore, and since $h([x', y'])$ is p-link-symmetric with center b_k, there exists a unique p-point y such that the arc $[x, y]$ is p-symmetric with center b_k. Also, $h(y') \in K_y$ and $L_p(y) = L_p(x)$, so $L_p(y) = L_q(y') + M$. This proves that for every q-point $x' \in E_q$, $L_q(x') > 0$, $x' \prec s'_{k+2}$, we have $u = h(x') \in \ell_p^{s_r+M}$, where $r = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_r+M}$ contains a p-point x such that $L_p(x) = r + M$.

Next $h([s'_{k+1}, s'_{k+2}]) = [a_{k+1}, a_{k+2}]$, $s_{M+k+1} \in K_{a_{k+1}}$ and $s_{M+k+2} \in K_{a_{k+2}}$. Let the q-point $x'_{k+1} \in [s'_{k+2}, s'_{k+3}]$ be such that the arc $[s'_{k+1}, x'_{k+1}]$ is q-symmetric with center s'_{k+2}. Then $h([s'_{k+1}, x'_{k+1}])$ is p-link-symmetric with center s_{M+k+2}. Since there exists a unique

p-point b_{k+1} such that the arc $[s_{M+k+1}, b_{k+1}]$ is p-symmetric with center s_{M+k+2}, it follows that $h(x'_{k+1}) \in K_{b_{k+1}}$. Also, $L_q(x'_{k+1}) = k+1$ and $L_p(b_{k+1}) = M+k+1$. Since $[s'_{k+1}, x'_{k+1}]$ is q-symmetric with center s'_{k+2} and $[s_{M+k+1}, b_{k+1}]$ is p-symmetric with center s_{M+k+2}, the same argument as above shows that for every q-point $x' \in E_q$, $L_q(x') > 0$, $x' \prec x'_{k+1}$, we have $u = h(x') \in \ell_p^{s_{r+M}}$, where $r = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_{r+M}}$ contains a p-point x such that $L_p(x) = r + M$. This proves the induction step.

(2) Let x be a p-point such that $L_p(x) > 0$ and $v = h^{-1}(x)$ lies beyond the κ-th snappy g-point. Since h^{-1} is also a homeomorphism and $h^{-1}(\mathcal{C}_p) \prec \mathcal{C}_g$, (1) implies that there exists M' such that $v \in \ell_g^{s''_{r+M'}}$, where $r = L_p(x)$. Also the arc component $K_v \subset \ell_g^{s''_{r+M'}}$ contains a g-point x'' such that $L_g(x'') = r + M'$.

Let x' be a q-point such that $L_q(x') > 0$, x' lies beyond the κ-th snappy g-point and $u = h(x')$ lies beyond the κ-th snappy p-point. Then $u \in \ell_p^{s_{r'+M}}$, where $r' = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_{r'+M}}$ contains a p-point x such that $L_p(x) = r' + M$. Also $v = h^{-1}(x) \in \ell_g^{s''_{r'+M+M'}}$ and the arc component $K_v \subset \ell_g^{s''_{r'+M+M'}}$ contains a g-point x'' such that $L_g(x'') = L_q(x') + M + M'$. Since $h^{-1} \circ h = id$, we have $x'' = x'$. Also $L_g(x'') = L_q(x') + q - g$ implies that $M + M' = q - g$. Since the number of q-points in $[s'_i, s'_{i+1}]$ with q-level l, $l \in \mathbb{N}_0$, is the same as the number of g-points in $[s''_{q-g+i}, s''_{q-g+i+1}]$ with g-level $q - g + l$, it follows that this number is the same as the number of p-points in $[s_{M+i}, s_{M+i+1}]$ with p-level $M + l$. $\square$

Proof of Theorem 1.1. By [23] we can assume that the critical points of T_s and $T_{s'}$ have infinite orbits. Therefore the above proposition shows that

$$FP_q([s'_k, s'_{k+1}]) = FP_{p+M}([s_{M+k}, s_{M+k+1}]) = FP_p([s_k, s_{k+1}]),$$

for every positive integer k, and therefore $FP(\mathfrak{C}') = FP(\mathfrak{C})$, implying $s' = s$. This proves the Ingram Conjecture. $\square$

5   Pseudo-isotopy

Throughout this section, $h : \varprojlim([0,1], T_s) \to \varprojlim([0,1], T_s)$ will be an arbitrary self-homeomorphism. We will extend Proposition 4.2 in order to prove the result on pseudo-isotopy. Note that (1) and (2) of Proposition 4.2 together show that h induces an order preserving injection $h_{q,p}$ from E_q to E_p such that $h_{q,p}(E_{q,i}) = E_{p,M+i} = E_{p+M,i}$ for every $i \in \mathbb{N}_0$, where $E_{r,l}$ denotes the set of all r-points with r-level l (see Definition 2.5). In fact $h_{q,p}$ is an order preserving bijection from E_q to E_{p+M} and is defined as follows:

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Definition 5.1. Let $x \in E_q$. If $x = s_i'$ for some $i \in \mathbb{N}$, we define $h_{q,p}(s_i') = s_{M+i} \in E_p$. For all other $x \in E_q$, there exists $i \in \mathbb{N}$ such that $x \in (s_i', s_{i+1}')$. By Proposition 4.2, the number of q-points of (s_i', s_{i+1}') is the same as the number of $(p+M)$-points of (s_{M+i}, s_{M+i+1}'). Let $(s_i', s_{i+1}') \cap E_q = \{x^0, \ldots, x^n\}$ and $(s_{M+i}, s_{M+i+1}') \cap E_{p+M} = \{y^0, \ldots, y^n\}$. We define $h_{q,p}(x^i) = y^i$, $i = 0, \ldots, n$.

The next lemma shows that $h_{q,p}$ is essentially independent of q and p.

Lemma 5.2. If $q_1, p_1 \in \mathbb{N}$ are such that $h(\mathcal{C}_{q_1}) \prec \mathcal{C}_{p_1} \prec h(\mathcal{C}_q) \prec \mathcal{C}_p$, then $h_{q_1,p_1}|_{E_{q_1}} = h_{q,p}|_{E_{q_1}}$.

Proof. By Proposition 4.2, $h(\mathcal{C}_q) \prec \mathcal{C}_p$ implies that there exists $M \in \mathbb{Z}$ such that $h_{q,p}(E_{q,i}) = E_{p,M+i}$ for every $i \in \mathbb{N}_0$. Also, $h(\mathcal{C}_{q_1}) \prec \mathcal{C}_{p_1}$ implies that there exists $M_1 \in \mathbb{Z}$ such that $h_{q_1,p_1}(E_{q_1,i}) = E_{p_1,M_1+i}$ for every $i \in \mathbb{N}_0$. Let $r, l \in \mathbb{N}$ be such that $q_1 = q + r$ and $p_1 = p + l$. Since $E_{q+r,i} = E_{q,r+i}$, we have

$$h_{q,p}(E_{q+r,i}) = h_{q,p}(E_{q,r+i}) = E_{p,M+r+i},$$

and also

$$h_{q+r,p+l}(E_{q+r,i}) = E_{p+l,M_1+i} = E_{p,M_1+l+i}.$$

We want to prove that $M + r = M_1 + l$. To see this it suffices to pick a convenient point x in $E_{q+r,j}$ for some $j \in \mathbb{N}$, and to prove that $h_{q,p}(x) = y = h_{q+r,p+l}(x)$. Then the fact that $y \in E_{p,M+r+j}$ and $y \in E_{p,M_1+l+j}$ implies that $M + r + j = L_p(y) = M_1 + l + j$. For us, the convenient choice of $x \in E_{q+r} \subset E_q$ is a snappy $(q+r)$-point.

Let us denote the snappy $(q+r)$-points by $\hat{s}_i'$ and the snappy $(p+l)$-points by $\hat{s}_i$, while as before s_i' denotes the snappy q-points and s_i denotes the snappy p-points. Note that the snappy $(q+r)$-point $\hat{s}_i'$ is the same as the snappy q-point s_{i+r}', and the snappy $(p+l)$-point $\hat{s}_i$ is the same as the snappy p-point s_{i+l}. Let us denote the maximal $(q+r)$-link-symmetric arc with the center $\hat{s}_i'$ by $\hat{A}_i'$, and the maximal $(p+l)$-link-symmetric arc with the center $\hat{s}_i$ by $\hat{A}_i$, while as before A_i' denotes the maximal q-link-symmetric arc with the center s_i', and A_i denotes the maximal p-link-symmetric arc with the center s_i. Note that $h(\hat{A}_i') \subseteq \hat{A}_{M_1+i}$, $h(A_{i+r}') \subseteq A_{M+i+r}$ and $\hat{s}_i' = s_{i+r}'$. Also, the center of $\hat{A}_{M_1+i}$ is $\hat{s}_{M_1+i} = s_{M_1+i+l}$ and the center of A_{M+i+r} is s_{M+i+r}. Therefore, $s_{M+i+r} = s_{M_1+i+l}$ and $M + r = M_1 + l$. $\qquad\square$

Corollary 5.3. $R = M + p - q$ does not depend on M, p, q.

Proof. By Lemma 5.2, $M_1 + l = M + r$. Therefore $R_1 = M_1 + p_1 - q_1 = M_1 + (p+l) - (q+r) = M + r + p - q - r = R$. $\qquad\square$

Definition 5.4. We call an arc $B \in \mathfrak{C}$ a *p-bridge* if the boundary points of B are p-points with p-level 0, and if $L_p(x) \neq 0$ for every p-point $x \in \operatorname{Int} B$.

Corollary 5.5. Let $B' \subset \mathfrak{C}$ be a $(q+1)$-bridge and $\partial B' = \{a', b'\}$. There exists a $(p+M+1)$-bridge B such that for $\partial B = \{a, b\}$ we have $h(B') \subset K_a \cup B \cup K_b$ and $h(a') \in K_a$, $h(b') \in K_b$, where K_a and K_b are the arc-components of the link $\ell_p^{s_{M+1}}$ of $\mathcal{C}_p$ containing a and b respectively.

Proof. Proposition 4.2 dealt with points in $E_{q,j}$ for $j \geq 1$, but bridges involve points of level zero. Since $E_{q,1} = E_{q+1,0}$, in this corollary we can work with $(q+1)$-bridges.

For each $j \geq 1$, $E_{q,j}$ is contained in a single link $\ell_q^{s'_j} \in \mathcal{C}_q$ and by Proposition 4.2, for $\ell_p^{s_{M+j}} \supseteq h(\ell_q^{s'_j})$, every point of $h(E_{q,j})$ is contained in an arc component of $\ell_p^{s_{M+j}}$ which contains a p-point of $E_{p,M+j} = E_{p+M,j}$. Since $E_{q+1,0} = E_{q,1}$ and $E_{p+M+1,0} = E_{p+M,1}$, every point of $h(E_{q+1,0}) = h(E_{q,1})$ is contained in an arc component of $\ell_p^{s_{M+1}}$ which contains a point of $E_{p+M,1} = E_{p+M+1,0}$.

Every two adjacent points of $E_{q+1,0}$ are the boundary points of a $(q+1)$-bridge, and every two adjacent points of $E_{p+M+1,0}$ are the boundary points of a $(p+M+1)$-bridge. We also have $h_{q,p+M}(E_{q+1,0}) = h_{q,p+M}(E_{q,1}) = E_{p,M+1} = E_{p+M+1,0}$. Therefore, for every $(q+1)$-bridge B' there exists a $(p+M+1)$-bridge B such that $h_{q,p+M}(B') = B$. More precisely, for every $(q+1)$-bridge B' and $\partial B' = \{a', b'\}$, there exists a $(p+M+1)$-bridge B such that for $\partial B = \{a, b\}$ we have $h(B') \subset K_a \cup B \cup K_b$ with $h(a') \in K_a$ and $h(b') \in K_b$. Note that if B' is a $(q+1)$-bridge with center z' and $\partial B' = \{a', b'\}$ and B' is contained in a single link $\ell_{q+1}^{s'_1}$, then $h(B')$ is contained in the arc component $K_a = K_b$ which contains also a $(p+M+1)$-point z such that $L_{p+M+1}(z) = L_{q+1}(z')$. So the arc component K_a contains a $(p+M+1)$-bridge B with center z and we have again $h(B') \subset K_a \cup B \cup K_b$. $\square$

Example 5.6. A $\sin \frac{1}{x}$-*continuum* is a homeomorphic copy of

$$\left(\{0\} \times [-1, 1] \right) \cup \left\{ (x, \sin \frac{1}{x}) : x \in (0, 1] \right\}$$

and the arc $\{0\} \times [-1, 1]$ is called the *bar* of the $\sin \frac{1}{x}$-continuum. Assume that $s > \sqrt{2}$ is such that the inverse limit $\varprojlim([0, 1], T_s)$ contains a $\sin \frac{1}{x}$-continuum H. (Such s exist in abundance, cf. [2] and [13].) Then $\{\sigma^{-n}(H)\}_{n=0}^{\infty}$ is a sequence of pairwise disjoint $\sin \frac{1}{x}$-continua with $\operatorname{diam}(\sigma^{-n}(H)) \to 0$ as $n \to \infty$. There is then a sequence of disjoint neighborhoods U_n of $\sigma^{-n}(H)$ with $diam(U_n) \to 0$. For each n, $U_n \cap \mathfrak{C}$ contains arbitrarily long arcs. Pick a sequence of arcs $A_n \subset U_n \cap \mathfrak{C}$ of arc-length $\geq n + 1$, and construct a bijection $h : \varprojlim([0, 1], T_s) \circlearrowleft$ such that h is the identity on $\varprojlim([0, 1], T_s) \setminus \cup_n A_n$ and on

each A_n, h fixes ∂A_n, but moves some points in A_n homeomorphically such that there is $x_n \in A_n$ with $\bar{d}(x_n, h(x_n)) = n$. Since $\mathrm{diam}(U_n) \to 0$, we find that h is continuous and bijective. Finally the compactness of $\varprojlim([0,1], T_s)$ implies that h is a homeomorphism. Even though h is isotopic to the identity, $\sup_{x \in \mathfrak{C}} \bar{d}(x, h(x)) = \infty$.

Therefore we cannot assume that a general self-homeomorphism of $\varprojlim([0,1], T_s)$ has an $R \in \mathbb{Z}$ such that $\sup_x \bar{d}(h(x), \sigma^R(x)) < \infty$. Block et al. [8, Theorem 4.2] used this property to conclude that h and σ^R are pseudo-isotopic, *i.e.*, they permute the composants of $\varprojlim([c_2, c_1], T_s)$ in the same way. However, since $\sigma^{-R} \circ h$ preserves $(q+1)$-bridges for some $R \in \mathbb{Z}$ and q sufficiently large, we can still follow the argument from [8].

Proof of Theorem 1.2. Let $P = s/(1+s) > 1/2$ be the orientation reversing fixed point of T_s and Q the center between c_2 and c_1. Let $\varepsilon = \mathrm{mesh}(\mathcal{C}_p)$ in Definition 2.4. Without loss of generality, we can take $\varepsilon/2 < \min\{|c - P|, |c - Q|\}$. Let $x \in \varprojlim([0,1], T_s) \setminus \mathfrak{C} = \varprojlim([c_2, c_1], T_s)$ be arbitrary. Recall that the composant of x in $\varprojlim([c_2, c_1], T_s)$ is the union of all proper subcontinua of $\varprojlim([c_2, c_1], T_s)$ containing x. Without loss of generality we can fix $q \in \mathbb{N}$ such that $\pi_{q+1}(x) \geq P$. Fix $p \in \mathbb{N}$ and $M \in \mathbb{Z}$ as in Proposition 4.2 such that $h(\mathcal{C}_q) \preceq \mathcal{C}_p$ and h sends $(q+1)$-bridges to $(p + M + 1)$-bridges in terms of Corollary 5.5. Let $R = M + p - q$, so $p + M + 1 = q + R + 1$. Since by Corollary 5.3, R does not depend on q and p, we can take q and p larger than $|R|$ without loss of generality.

Recall that the links ℓ_p^k of $\mathcal{C}_p$ are of the form $\ell_p^k = \pi_p^{-1}(I_p^k)$ of width $\leq \varepsilon s^{-p}/2$. The map σ^{-R} maps the chain $\mathcal{C}_p$ to a chain $\tilde{\mathcal{C}}_{p-R}$ whose links are of the form $\pi_{p-R}^{-1}(I_p^k)$ and hence also with width $\leq \varepsilon s^{-p}/2$; this chain is coarser than $\mathcal{C}_{p-R}$ if $R < 0$. Furthermore, the σ^{-R}-image of a $(q + R + 1)$-bridge is a $(q+1)$-bridge.

Take $\tilde{h} = \sigma^{-R} \circ h$. Since $h(\mathcal{C}_q) \preceq \mathcal{C}_p$, we have $\tilde{h}(\mathcal{C}_q) \preceq \tilde{\mathcal{C}}_{p-R}$ and $\tilde{h}$ sends $(q+1)$-bridges to $(q+1)$-bridges, but the 'error' allowed in Corollary 5.5, *i.e.*, the arc-components of links from $\mathcal{C}_p$, must now be replaced by arc-components of links of $\tilde{\mathcal{C}}_{p-R}$. Recall that $\mathrm{width}(\mathcal{C}_p) = \max_j |I_p^j|$, and $|\pi_{p-i}(\ell_p^j)| = |\pi_p(\ell_p^j)|s^i = |I_p^j|s^i$, for every $0 \leq i \leq p$. Therefore, $\pi_{p-R}(\tilde{\ell}_{p-R}^j) = \pi_{q-M}(\tilde{\ell}_{q-M}^j) \leq \varepsilon s^{-p}/2$, and $\pi_{q+1}(\tilde{\ell}_{p-R}^j) = \pi_{q+1}(\tilde{\ell}_{q-M}^j) = \pi_{q-M}(\tilde{\ell}_{q-M}^j)s^{-M-1} \leq \varepsilon s^{-p-M-1}/2$. Thus, the $(q+1)$-th projection of links of $\tilde{\mathcal{C}}_{p-R}$ are intervals of length $\leq \varepsilon s^{-(p+M+1)}/2 = \varepsilon s^{-(q+R+1)}/2$, see Figure 6.

The $(q+1)$-bridges that are small enough to belong to one or two links of $\mathcal{C}_q$ will map to arcs contained in the link $\tilde{\ell}_{p-R}$. Since $\pi_{q+1}(x) \geq P$ and $\varepsilon s^{-(q+1)}/2 < |c - P|$, no such short bridge can be close to x. On the longer $(q+1)$-bridges of $\mathcal{C}_q$ that map outside of $\tilde{\ell}_{p-R}$, $\tilde{h}$ acts as a trivial one-to-one correspondence, sending the first such bridge to the first, the second to the second, etc.
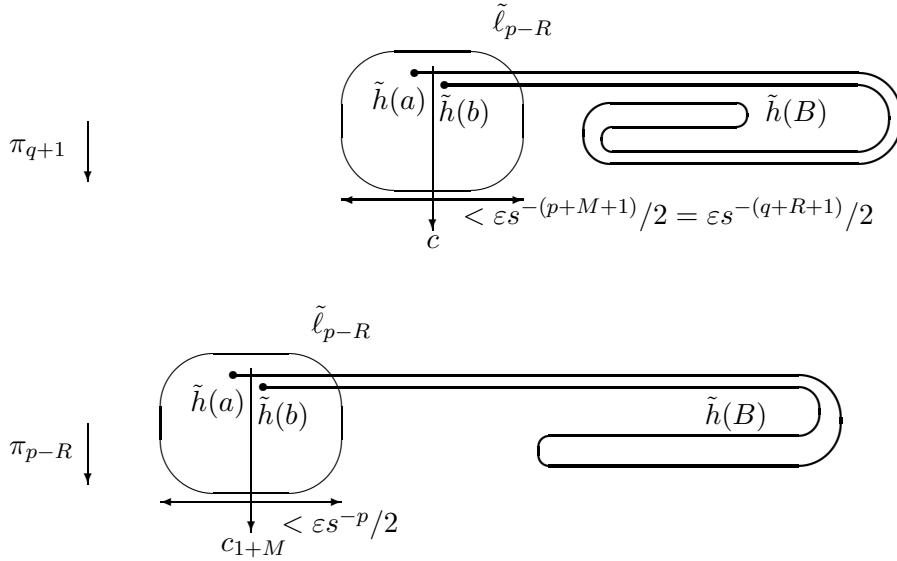
Figure 6: The $(p-R)$-th and $(q+1)$-th projection of 'the bridge' $\tilde{h}(B)$ with relevant link $\tilde{\ell}_{p-R}$. The picture is suggestive of $M+1 \leq 0$; if instead $M+1 > 0$, then $\tilde{h}(B)$ contains fewer $(q+1)$-points than $(p-R)$-points.

Find a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathfrak{C}$ such that $x_n \to x$. Then for large n, x_n belongs to a long $(q+1)$-bridge, and by the above argument, $\tilde{h}(x_n)$ and x_n belong to the same $(q+1)$-bridge up to an 'error' of at most $\varepsilon s^{-(q+R+1)}/2$. Take $H_n = [\tilde{h}(x_n), x_n]$ and a subsequence such that $H_{n_j} \to H$ in Hausdorff topology. Clearly H is a continuum and $x, \tilde{h}(x) \in H$. Since $\pi_{q+1}(x) \geq P$, the arcs H_{n_j} belong to arcs whose $(q+1)$-projections belong to $[c - \varepsilon s^{-(q+R+1)}/2, c_1]$ for all sufficiently large j. Since $q + R + 1 \geq 1$ and $\varepsilon/2 < c - Q$, we have $Q < c - \varepsilon/2 < c - \varepsilon s^{-(q+R+1)}/2$ implying $[c - \varepsilon s^{-(q+R+1)}/2, c_1] \subset [Q, c_1]$.

Therefore $\pi_{q+1}(H_{n_j}), \pi_{q+1}(H) \subset [Q, c_1]$, and since $[Q, c_1]$ is a proper subset of $[c_2, c_1]$ and the inclusion holds for arbitrarily large q, H is a proper subcontinuum of $\varprojlim([c_2, c_1], T_s)$. It follows that $\tilde{h}(x)$ and x belong to the same composant of $\varprojlim([c_2, c_1], T_s)$. Apply σ^R to find that $h(x)$ and $\sigma^R(x)$ belong to the same composant as well. $\qquad \square$

Pseudo-isotopy of h implies that the number of composants being mapped to themselves is the same for h^n and σ^{nR}. This number grows like s^{nR}, which in [9] provides a proof of the Ingram conjecture for tent maps with periodic critical point. In this situation, [9] in fact also shows that h is isotopic to a power of the shift. Due to the existence of composants that are not arc-connected, this is not so clear in the general case.

Remark 5.7. Not every pseudo-isotopy is an isotopy. For instance, a homeomorphism

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flipping the bar of a $\sin \frac{1}{x}$-continuum cannot be isotopic to the identity. If the bonding map is a quadratic map within the first period doubling cascade, then the inverse limit space is a finite collection of $\sin \frac{1}{x}$-continua, see [7], and we can indeed construct homeomorphism that are pseudo-isotopic but not isotopic to the identity. Among those tent maps T_s, $s \in [\sqrt{2}, 2]$, whose inverse limit space is known to contain $\sin \frac{1}{x}$-continua, both in [2] and [13], the topology is much more complicated, as more than a single ray can be expected to accumulate on their bars. Thus the following question is very relevant:

Is every self-homeomorphism of $\varprojlim([0, 1], T_s)$ isotopic to a power of the shift?

We know this to be true if c is periodic or non-recurrent [9, 10], but this case is simpler, because the only proper subcontinua of $\varprojlim([c_2, c_1], T_s)$ are arcs or points.

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